REGULAR FLIP EQUIVALENCE OF SURFACE TRIANGULATIONS

SIMON A. KING

Abstract. Any two triangulations of a closed surface with the same number of vertices can be transformed into each other by a sequence of flips, provided the number of vertices exceeds a number N depending on the surface. Examples show that in general N is bigger than the minimal number of vertices of a triangulation. The existence of N was known, but no estimate. This paper provides an estimate for N that is linear in the Euler characteristic of the surface.

1. Results on flip equivalence

Let F be a closed surface and let $\chi(F)$ be its Euler characteristic. A **singular triangulation** of F is a graph T embedded in F such that each face of $F \setminus T$ is bounded by an edge path of length three. We denote by $v(T)$, $e(T)$ and $f(T)$ the number of vertices, edges and faces of T. If T is without loops and multiple edges and has more than three faces, then T corresponds to a triangulation of F in the classical meaning of the word; in order to avoid confusions, we use for it the term **regular triangulation**.

Let e be an edge of a singular triangulation T and suppose that there are two distinct faces δ_1 and δ_2 adjacent to e. The faces δ_1 and δ_2 form a (possibly degenerate) quadrilateral, containing e as a diagonal. A **flip** of T along e replaces e by the opposite diagonal of this quadrilateral, see Figure 1. The flip is called **regular**, if both T and the result of the flip are regular triangulations. Two singular (resp. regular) triangulations T_1, T_2 of a closed surface are called flip equivalent (resp. regularly flip equivalent), if they are related by a finite sequence of flips (resp. regular flips) and isotopy.

The following result is well known, and there are many proofs for it. There are interesting applications to the automatic structure of mapping class groups, see [4] or [8].

Lemma 1. *Any two singular triangulations T_1 and T_2 of a closed surface F with $v(T_1) = v(T_2)$ are flip equivalent.* □

One might ask whether any two regular triangulations of F with the same number of vertices are *regularly* flip equivalent. The answer is "Yes" in special cases: any two regular triangulations of the sphere [9], the torus [2], the projective plane or the Klein bottle [6] with the same number of vertices are regularly flip equivalent. But in general, the answer is "No": it is known that there are 59 different triangulations of the closed oriented surface of genus six based on the complete graph with 12 vertices, see [1]. Such a triangulation does not admit any regular flip, thus the different triangulations are not regularly flip equivalent.

This paper is devoted to the proof of the following theorem. A preliminary version of this paper has been appeared in [3].

Theorem 1. *Let F be a closed surface and $N(F) = 9450 - 6020\chi(F)$. Any two regular triangulations T_1 and T_2 of F with $v(T_1) = v(T_2) \geq N(F)$ are regularly flip equivalent.*

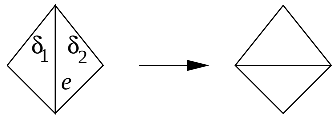

FIGURE 1. A flip

Negami [7] stated the mere existence of $N(F)$ without an estimate. The estimate in Theorem 1 is far from being best possible, at least for the surfaces up to genus one. The number $N(F)$ is negative if and only if F is a sphere, in which case the statement is true since the transformation by regular flips is always possible, by Wagner's Theorem [9]. We assume in the following that F is not the sphere.

2. Proof of the Theorem

We need some additional notions. A **contraction** of a regular triangulation T along an edge e shrinks e to a vertex and eliminates the two faces adjacent to e, see Figure 2. The edge e is called **contractible** if the result of the contraction is still

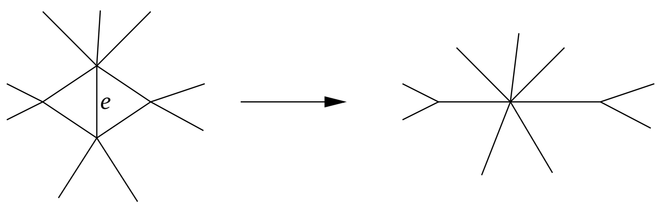

FIGURE 2. Contraction along an edge

a regular triangulation. A regular triangulation T is called **irreducible** if it does not contain contractible edges. The number of vertices of irreducible triangulations is bounded by the following result of Nakamoto and Ota [5].

Proposition 1. *If T is an irreducible triangulation of a closed surface F which is not the sphere, then $v(T) \leq 270 - 171\chi(F)$.* □

Let δ be a face of a regular triangulation T. A **face subdivision** of T along δ replaces δ by the cone over its boundary, see Figure 3, and the result is denoted $s_\delta T$. If δ and δ' are two faces of T, then $s_\delta T$ and $s_{\delta'} T$ are regularly flip equivalent, which is easy to see. Let T_1 and T_2 be regular triangulations and δ_1, δ_2 faces of T_1, T_2. It follows that if T_1 and T_2 regularly flip equivalent, then so are $s_{\delta_1} T_1$ and $s_{\delta_2} T_2$. If T_2 is obtained from T_1 by m successive face subdivisions, then we write $T_2 = s^m(T_1)$. The notation is ambiguous, but by the preceding remark only up to regular flip equivalence. After these preliminaries, we can cite a lemma of Negami [7].

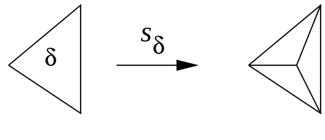

FIGURE 3. A face subdivision

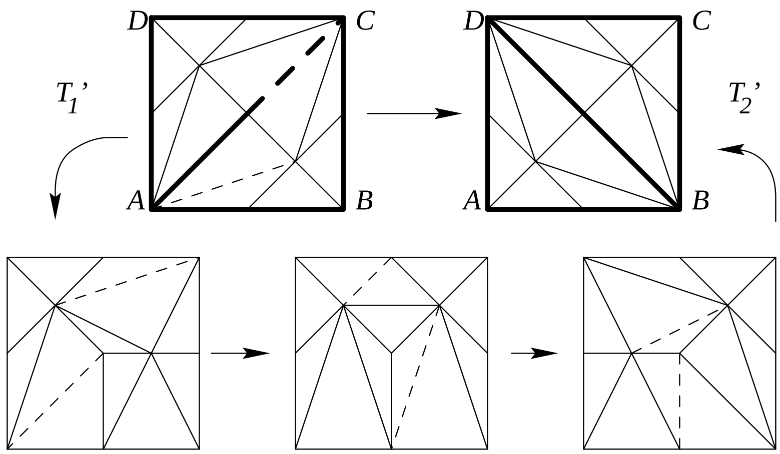

FIGURE 4. Flips in the barycentric subdivision

Lemma 2. *Let T_1 and T_2 be regular triangulations of F. If T_2 is obtained by contracting some edges of T_1, then T_1 is regularly flip equivalent to $s^m(T_2)$, with $m = v(T_1) - v(T_2)$.* □

Let T' denote the barycentric subdivision of a singular triangulation T of F.

Lemma 3. *Let T_1 and T_2 be two singular triangulations of F with $v(T_1) = v(T_2)$. Then T_1'' and T_2'' are regularly flip equivalent.*

Proof. It is easy to verify that T_1'' and T_2'' are regular triangulations of F. By Proposition 1, we know that T_1 and T_2 are related by not necessarily regular flips. Let T_2 be obtained from T_1 by a single flip. Then T_1' can be transformed into T_2' by the sequence of flips and isotopies that is explicitly given in Figure 4. The edges of T_i are drawn bold, and the edges of T_i' under flip are dotted. None of these flips introduces a loop. It is possible that some flip for T_i' introduces a multiple edge. This happens only if some of the vertices A, B, C and D of T_1 coincide. We iterate the construction, i.e., we replace each flip for T_i' by a flip sequence for T_i''. Since the four vertices of T_1' of each quadrilateral involved in a flip are pairwise distinct, none of these flips introduces a loop or a multiple edge, thus all flips are regular. □ □

Corollary 1. *Let T_1 and T_2 be two regular triangulations of F with $v(T_1) = v(T_2)$. Then $s^m(T_1)$ and $s^m(T_2)$ are regularly flip equivalent, with $m = 35\,(v(T_1) - \chi(F))$.*

Proof. For any singular triangulation T of F, we have $v(T') = v(T) + e(T) + f(T)$. Since $2e(T) = 3f(T)$, we obtain $f(T) = 2(v(T) - \chi(F))$ and $v(T') = 6v(T) - 5\chi(F)$. It follows easily that $v(T_i'') - v(T_i) = m$ for $i = 1, 2$.

One obtains T_i'' from T_i by m face subdivisions and some regular flips, see Figure 5 for the first barycentric subdivision. The figure shows the neighbourhood of a face, and the edges under flip are dotted. So $s^m(T_1) \sim T_1'' \sim T_2'' \sim s^m(T_2)$ by the preceding Lemma. □ □

Now we finish the proof of Theorem 1. Let T_1, T_2 be two regular triangulations of F with $v(T_1) = v(T_2) = M \geq N(F)$ with

$$N(F) = 35\,((270 - 171\chi(F)) - \chi(F)) = 9450 - 6020\chi(F).$$

By contractions along some edges, T_i ($i \in \{1, 2\}$) can be transformed into an irreducible triangulation S_i. By Lemma 2, T_i is regularly flip equivalent to $s^{M-v(S_i)}S_i$. By Proposition 1 and Corollary 1, $s^{N(F)-v(S_1)}S_1$ and $s^{N(F)-v(S_2)}S_2$ are regularly

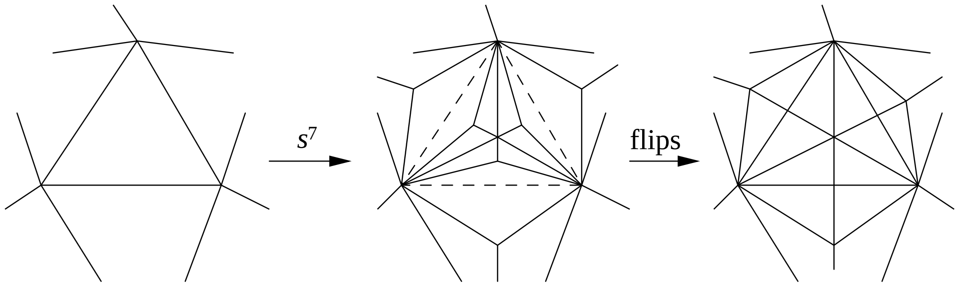

FIGURE 5. How to obtain the barycentric subdivision by flips and face subdivisions

flip equivalent, and so are also $s^{M-v(S_1)}S_1$ and $s^{M-v(S_2)}S_2$ after further face subdivisions. Therefore also T_1 and T_2 are regularly flip equivalent. **q.e.d.**

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SIMON A. KING, DEPARTMENT OF MATHEMATICS, DARMSTADT UNIVERSITY OF TECHNOLOGY, SCHLOSSGARTENSTR. 7, 64289 DARMSTADT, GERMANY

E-mail address: `king@mathematik.tu-darmstadt.de`